

第八届全国大学生数学竞赛决赛试题参考答案

(非数学类, 2017 年)

一、填空题

1. 过单叶双曲面 $\frac{x^2}{4} + \frac{y^2}{2} - 2z^2 = 1$ 与球面 $x^2 + y^2 + z^2 = 4$ 的交线且与直线 $\begin{cases} x=0 \\ 3y+z=0 \end{cases}$ 垂直的

平面方程为 _____.

答案: $y - 3z = 0$.

2. 设可微函数 $f(x, y)$ 满足 $\frac{\partial f}{\partial x} = -f(x, y)$, $f\left(0, \frac{\pi}{2}\right) = 1$, 且 $\lim_{n \rightarrow \infty} \left(\frac{f\left(0, y + \frac{1}{n}\right)}{f(0, y)} \right)^n = e^{\cot y}$, 则

$f(x, y) =$ _____.

答案: $f(x, y) = e^{-x} \sin y$.

3. 已知 A 为 n 阶可逆反对称矩阵, b 为 n 元列向量, 设 $B = \begin{pmatrix} A & b \\ b^T & 0 \end{pmatrix}$, 则 $\text{rank}(B) =$ _____.

答案: n .

4. $\sum_{n=1}^{100} n^{-\frac{1}{2}}$ 的整数部分为 _____.

答案: 18.

5. 曲线 $L_1: y = \frac{1}{3}x^3 + 2x (0 \leq x \leq 1)$ 绕直线 $L_2: y = \frac{4}{3}x$ 旋转所生成的旋转曲面的面积为

_____.

答案: $\frac{\sqrt{5}(2\sqrt{2}-1)}{3}\pi$.

二、设 $0 < x < \frac{\pi}{2}$, 证明: $\frac{4}{\pi^2} < \frac{1}{x^2} - \frac{1}{\tan^2 x} < \frac{2}{3}$.

证 设 $f(x) = \frac{1}{x^2} - \frac{1}{\tan^2 x} \left(0 < x < \frac{\pi}{2} \right)$, 则

$$f'(x) = -\frac{2}{x^3} + \frac{2 \cos x}{\sin^3 x} = \frac{2(x^3 \cos x - \sin^3 x)}{x^3 \sin^3 x}, \quad (1)$$

令 $\varphi(x) = \frac{\sin x}{\sqrt[3]{\cos x}} - x \left(0 < x < \frac{\pi}{2} \right)$, 则

$$\varphi'(x) = \frac{\cos^{4/3} x + \frac{1}{3} \cos^{-2/3} x \sin^2 x}{\cos^{2/3} x} - 1 = \frac{2}{3} \cos^{2/3} x + \frac{1}{3} \cos^{-4/3} x - 1$$

由均值不等式, 得

$$\frac{2}{3} \cos^{2/3} x + \frac{1}{3} \cos^{-4/3} x = \frac{1}{3} (\cos^{2/3} x + \cos^{2/3} x + \cos^{-4/3} x) > \sqrt[3]{\cos^{2/3} x \cdot \cos^{2/3} x \cdot \cos^{-4/3} x} = 1,$$

所以当 $0 < x < \frac{\pi}{2}$ 时, $\varphi'(x) > 0$, 从而 $\varphi(x)$ 单调递增, 又 $\varphi(0) = 0$, 因此 $\varphi(x) > 0$, 即

$$x^3 \cos x - \sin^3 x < 0.$$

由 (1) 式得 $f'(x) < 0$, 从而 $f(x)$ 在区间 $\left(0, \frac{\pi}{2}\right)$ 单调递减. 由于

$$\lim_{x \rightarrow \frac{\pi}{2}^-} f(x) = \lim_{x \rightarrow \frac{\pi}{2}^-} \left(\frac{1}{x^2} - \frac{1}{\tan^2 x} \right) = \frac{4}{\pi^2}$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan^2 x} \right) = \lim_{x \rightarrow 0^+} \left(\frac{\tan x + x}{x} \cdot \frac{\tan x - x}{x \tan^2 x} \right) = 2 \lim_{x \rightarrow 0^+} \frac{\tan x - x}{x^3} = \frac{2}{3},$$

所以 $0 < x < \frac{\pi}{2}$ 时, 有 $\frac{4}{\pi^2} < \frac{1}{x^2} - \frac{1}{\tan^2 x} < \frac{2}{3}$.

三、设 $f(x)$ 为 $(-\infty, +\infty)$ 上连续的周期为 1 的周期函数, 且满足 $0 \leq f(x) \leq 1$ 与 $\int_0^1 f(x) dx = 1$.

证明: 当 $0 \leq x \leq 13$ 时, 有

$$\int_0^{\sqrt{x}} f(t) dt + \int_0^{\sqrt{x+27}} f(t) dt + \int_0^{\sqrt{13-x}} f(t) dt \leq 11,$$

并给出取等号的条件.

证 由条件 $0 \leq f(x) \leq 1$, 有

$$\int_0^{\sqrt{x}} f(t) dt + \int_0^{\sqrt{x+27}} f(t) dt + \int_0^{\sqrt{13-x}} f(t) dt \leq \sqrt{x} + \sqrt{x+27} + \sqrt{13-x}.$$

利用柯西不等式, 即: $\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \sum_{i=1}^n a_i^2 \cdot \sum_{i=1}^n b_i^2$, 等号当 a_i 与 b_i 对应成比例时成立. 有

$$\sqrt{x} + \sqrt{x+27} + \sqrt{13-x} = 1 \cdot \sqrt{x} + \sqrt{2} \cdot \sqrt{\frac{1}{2}(x+27)} + \sqrt{\frac{2}{3}} \cdot \sqrt{\frac{3}{2}(13-x)}$$

$$\leq \sqrt{1+2+\frac{2}{3}} \cdot \sqrt{x+\frac{1}{2}(x+27)+\frac{3}{2}(13-x)}=11.$$

且等号成立的充分必要条件是:

$$\sqrt{x}=\frac{3}{2}\sqrt{13-x}=\frac{1}{2}\sqrt{x+27}, \text{ 即 } x=9.$$

所以

$$\int_0^{\sqrt{x}} f(t)dt + \int_0^{\sqrt{x+27}} f(t)dt + \int_0^{\sqrt{13-x}} f(t)dt \leq 11.$$

特别当 $x=9$ 时, 有

$$\int_0^{\sqrt{x}} f(t)dt + \int_0^{\sqrt{x+27}} f(t)dt + \int_0^{\sqrt{13-x}} f(t)dt = \int_0^3 f(t)dt + \int_0^6 f(t)dt + \int_0^2 f(t)dt$$

根据周期性, 以及 $\int_0^1 f(x)dx=1$, 有

$$\int_0^3 f(t)dt + \int_0^6 f(t)dt + \int_0^2 f(t)dt = 11 \int_0^1 f(t)dt = 11,$$

所以取等号的充分必要条件是 $x=9$.

四、设函数 $f(x, y, z)$ 在区域 $\Omega = \{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}$ 上具有连续的二阶偏导数, 且满

足 $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \sqrt{x^2 + y^2 + z^2}$. 计算 $I = \iiint_{\Omega} \left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} \right) dx dy dz$.

解 记球面 $\Sigma: x^2 + y^2 + z^2 = 1$ 外侧的单位法向量为 $\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$, 则

$$\frac{\partial f}{\partial n} = \frac{\partial f}{\partial x} \cos\alpha + \frac{\partial f}{\partial y} \cos\beta + \frac{\partial f}{\partial z} \cos\gamma.$$

考虑曲面积分等式:

$$\oiint_{\Sigma} \frac{\partial f}{\partial n} dS = \oiint_{\Sigma} (x^2 + y^2 + z^2) \frac{\partial f}{\partial n} dS. \quad (1)$$

对两边都利用高斯公式, 得

$$\oiint_{\Sigma} \frac{\partial f}{\partial n} dS = \oiint_{\Sigma} \left(\frac{\partial f}{\partial x} \cos\alpha + \frac{\partial f}{\partial y} \cos\beta + \frac{\partial f}{\partial z} \cos\gamma \right) dS = \iiint_{\Omega} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) dv \quad (2)$$

$$\begin{aligned} \oiint_{\Sigma} (x^2 + y^2 + z^2) \frac{\partial f}{\partial n} dS &= \oiint_{\Sigma} (x^2 + y^2 + z^2) \left(\frac{\partial f}{\partial x} \cos\alpha + \frac{\partial f}{\partial y} \cos\beta + \frac{\partial f}{\partial z} \cos\gamma \right) dS \\ &= 2 \iiint_{\Omega} \left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} \right) dv + \iiint_{\Omega} (x^2 + y^2 + z^2) \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) dv \end{aligned} \quad (3)$$

将 (2)、(3) 代入 (1) 并整理得

$$I = \frac{1}{2} \iiint_{\Omega} (1 - (x^2 + y^2 + z^2)) \sqrt{x^2 + y^2 + z^2} dv = \frac{1}{2} \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \int_0^1 (1 - \rho^2) \rho^3 d\rho = \frac{\pi}{6}.$$

五、设 n 阶方阵 A, B 满足 $AB = A + B$, 证明: 若存在正整数 k , 使 $A^k = O$ (O 为零矩阵), 则行列式 $|B + 2017A| = |B|$.

证 由条件 $A^k = O$ 可得 $|A| = 0$. 因为 $AB = A + B$, 所以

$$|B + 2017A| = |AB + 2016A| = |A| |B + 2016E| = 0.$$

又由 $B = A(B - E)$, 得 $|B| = 0$, 因此 $|B + 2017A| = |B|$.

六、设 $a_n = \sum_{k=1}^n \frac{1}{k} - \ln n$.

(1) 证明: 极限 $\lim_{n \rightarrow \infty} a_n$ 存在;

(2) 记 $\lim_{n \rightarrow \infty} a_n = C$, 讨论级数 $\sum_{n=1}^{\infty} (a_n - C)$ 的敛散性.

解 (1) 利用不等式: 当 $x > 0$ 时, $\frac{x}{1+x} < \ln(1+x) < x$, 有

$$a_n - a_{n-1} = \frac{1}{n} - \ln \frac{n}{n-1} = \frac{1}{n} - \ln \left(1 + \frac{1}{n-1} \right) \leq \frac{1}{n} - \frac{\frac{1}{n-1}}{1 + \frac{1}{n-1}} = 0,$$

$$\begin{aligned} a_n &= \sum_{k=1}^n \frac{1}{k} - \sum_{k=2}^n \ln \frac{k}{k-1} = 1 + \sum_{k=2}^n \left(\frac{1}{k} - \ln \frac{k}{k-1} \right) \\ &= 1 + \sum_{k=2}^n \left[\frac{1}{k} - \ln \left(1 + \frac{1}{k-1} \right) \right] \geq 1 + \sum_{k=2}^n \left[\frac{1}{k} - \frac{1}{k-1} \right] = \frac{1}{n} > 0, \end{aligned}$$

所以 $\{a_n\}$ 单调减少有下界, 故 $\lim_{n \rightarrow \infty} a_n$ 存在.

(2) 显然, 以 a_n 为部分和的级数为 $1 + \sum_{n=2}^{\infty} \left(\frac{1}{n} - \ln n + \ln(n-1) \right)$, 则该级数收敛于 C , 且

$a_n - C > 0$. 用 r_n 记该级数的余项, 则

$$a_n - C = -r_n = - \sum_{k=n+1}^{\infty} \left(\frac{1}{k} - \ln k + \ln(k-1) \right) = \sum_{k=n+1}^{\infty} \left(\ln \left(1 + \frac{1}{k-1} \right) - \frac{1}{k} \right).$$

根据泰勒公式, 当 $x > 0$ 时, $\ln(1+x) > x - \frac{x^2}{2}$, 所以

$$a_n - C > \sum_{k=n+1}^{\infty} \left(\frac{1}{k-1} - \frac{1}{2(k-1)^2} - \frac{1}{k} \right).$$

记 $b_n = \sum_{k=n+1}^{\infty} \left(\frac{1}{k-1} - \frac{1}{2(k-1)^2} - \frac{1}{k} \right)$, 下面证明正项级数 $\sum_{n=1}^{\infty} b_n$ 发散. 因为

$$c_n \triangleq n \sum_{k=n+1}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} - \frac{1}{2(k-1)(k-2)} \right) < nb_n < n \sum_{k=n+1}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} - \frac{1}{2k(k-1)} \right) = \frac{1}{2},$$

而当 $n \rightarrow \infty$ 时, $c_n = \frac{n-2}{2(n-1)} \rightarrow \frac{1}{2}$, 所以 $\lim_{n \rightarrow \infty} nb_n = \frac{1}{2}$. 根据比较判别法可知, 级数 $\sum_{n=1}^{\infty} b_n$ 发散.

因此, 级数 $\sum_{n=1}^{\infty} (a_n - C)$ 发散.