



§ 4.2 关系的运算

一、关系的定义域与值域

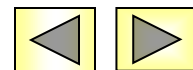
关系 R 的定义域: $dom R = \{ x \mid \exists y (< x, y > \in R) \}$,

即 R 中所有有序对的第一元素构成的集合。

关系 R 的值域: $ran R = \{ y \mid \exists x (< x, y > \in R) \}$,

即 R 中所有有序对的第二元素构成的集合。

关系 R 的域: $fld R = dom R \cup ran R$



§ 4.2 关系的运算

例5: 分别求出下列关系的定义域和值域。

$$(1) R_1 = \{ \langle x, y \rangle \mid x, y \in \mathbb{Z} \wedge x \leq y \};$$

$$(2) R_2 = \{ \langle x, y \rangle \mid x, y \in \mathbb{Z} \wedge x^2 + y^2 = 1 \};$$

$$(3) R_3 = \{ \langle x, y \rangle \mid x, y \in \{1, 2, 3, 4\} \wedge x > y \}$$

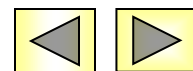
解: (1) $dom R_1 = ran R_1 = \mathbb{Z}$

$$(2) R_2 = \{ \langle 0, 1 \rangle, \langle 0, -1 \rangle, \langle 1, 0 \rangle, \langle -1, 0 \rangle \}$$

$$dom R_2 = ran R_2 = \{ 0, 1, -1 \}$$

$$(3) R_3 = \{ \langle 2, 1 \rangle, \langle 3, 1 \rangle, \langle 4, 1 \rangle, \langle 3, 2 \rangle, \langle 4, 2 \rangle, \langle 4, 3 \rangle \}$$

$$dom R_3 = \{ 2, 3, 4 \}, \quad ran R_3 = \{ 1, 2, 3 \}$$



§ 4.2 关系的运算

二、关系的常用运算

1、**逆**： F 是任意关系， F 的逆 $F^{-1} = \{ \langle x, y \rangle \mid yFx \}$

2、**合成**：任意两个关系 F 与 G 的合成，记作： $F \circ G$

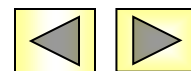
$$F \circ G = \{ \langle x, y \rangle \mid \exists z (xGz \wedge zFy) \}$$

3、**限制**：关系 F 在集合 A 上的限制，记作： $F \mid A$

$$F \mid A = \{ \langle x, y \rangle \mid xFy \wedge x \in A \}$$

4、**象**：集合 A 在 F 下的象，记作： $F[A]$

$$F[A] = \text{ran}(F \mid A)$$



§ 4.2 关系的运算

例6: 设 F, G 是 N 上的关系, 其定义为

$$F = \{ \langle x, y \rangle \mid x, y \in N \wedge y = x^2 \}$$

$$G = \{ \langle x, y \rangle \mid x, y \in N \wedge y = x + 1 \}$$

求 $G^{-1}, F \circ G, G \circ F, F \upharpoonright \{1, 2\}, F \upharpoonright \{1, 2\}$ 。

解: $G^{-1} = \{ \langle y, x \rangle \mid x, y \in N \wedge y = x + 1 \}$

列出 G^{-1} 中的元素即为

$$G^{-1} = \{ \langle 1, 0 \rangle, \langle 2, 1 \rangle, \langle 3, 2 \rangle, \dots, \langle x + 1, x \rangle, \dots \}$$



§ 4.2 关系的运算

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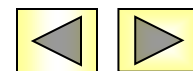
解: 为了求 $F \circ G$ 可以先直观表示如下:

对任何 $x \in N$

$$x \xrightarrow{G} x + 1 = z \xrightarrow{F} z^2 = y$$

$$\text{即 } y = (x + 1)^2$$

$$\text{因此 } F \circ G = \{ \langle x, y \rangle \mid x, y \in N \wedge y = (x + 1)^2 \}$$



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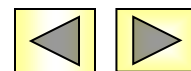
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求 $G^{-1}, F \circ G, G \circ F, F \upharpoonright \{1, 2\}, F[\{1, 2\}]$ 。

解: $F \upharpoonright \{1, 2\} = \{ \langle 1, 1 \rangle, \langle 2, 4 \rangle \}$

$$F[\{1, 2\}] = \text{ran}(F \upharpoonright \{1, 2\}) = \{ 1, 4 \}$$



§ 4.2 关系的运算

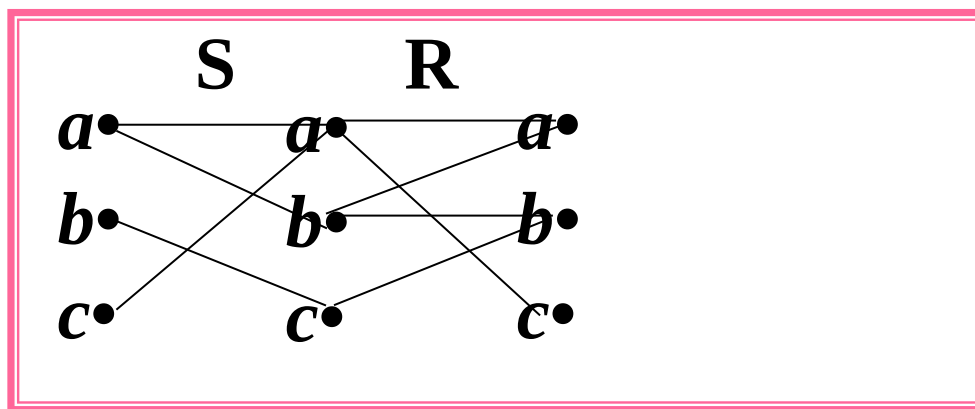
例7: 设集合 $A = \{a, b, c\}$, A 上的关系

$$R = \{\langle a, a \rangle, \langle a, c \rangle, \langle b, a \rangle, \langle b, b \rangle, \langle c, b \rangle\},$$

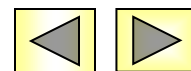
$$S = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, c \rangle, \langle c, a \rangle\},$$

求: $R \circ S$, $S \circ R$.

解:



$$R \circ S = \{\langle a, a \rangle, \langle a, b \rangle, \langle a, c \rangle, \langle b, b \rangle, \langle c, a \rangle, \langle c, c \rangle\}$$



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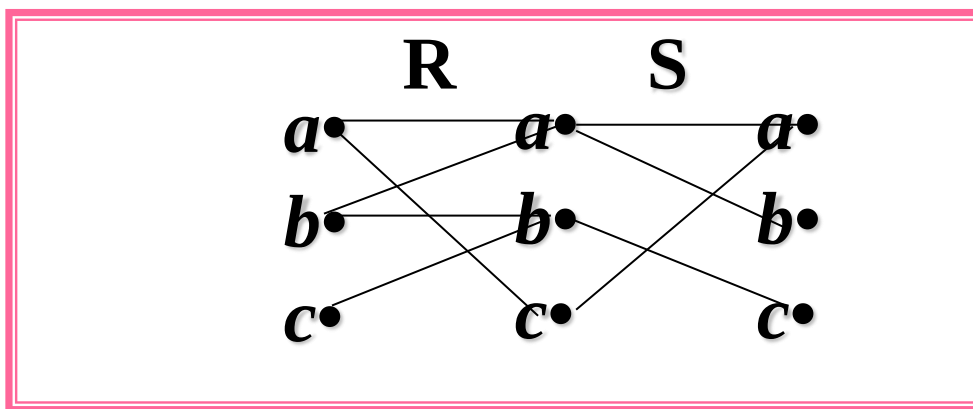
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求: $R \circ S$, $S \circ R$.

解:



$$R \circ S = \{\langle a, a \rangle, \langle a, b \rangle, \langle a, c \rangle, \langle b, b \rangle, \langle c, a \rangle, \langle c, c \rangle\}$$

$$S \circ R = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, a \rangle, \langle b, b \rangle, \langle b, c \rangle, \langle c, c \rangle\}$$



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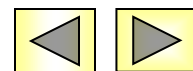
$$S = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, c \rangle, \langle c, a \rangle\},$$

求: $R \circ S$, $S \circ R$.

解:

$$M_R = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad M_S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_{R \circ S} = M_S \cdot M_R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad R \circ S = \{\langle a, a \rangle, \langle a, b \rangle, \langle a, c \rangle, \langle b, b \rangle, \langle c, a \rangle, \langle c, c \rangle\}$$



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例7: 设集合 $A = \{a, b, c\}$, A 上的关系

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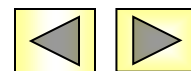
$$S = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, c \rangle, \langle c, a \rangle\},$$

求: $R \circ S$, $S \circ R$.

解:

$$M_R = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad M_S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_{S \circ R} = M_R \cdot M_S = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad S \circ R = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, a \rangle, \langle b, b \rangle, \langle b, c \rangle, \langle c, c \rangle\}$$



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三、关系运算的性质

1、 $(F^{-1})^{-1} = F$

2、 $\text{dom } F^{-1} = \text{ran } F$, $\text{ran } F^{-1} = \text{dom } F$

3、 $(F \circ G) \circ H = F \circ (G \circ H)$

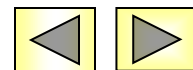
4、 $(F \circ G)^{-1} = G^{-1} \circ F^{-1}$

5、 $F \circ (G \cup H) = (F \circ G) \cup (F \circ H)$

6、 $F \circ (G \cap H) \subseteq (F \circ G) \cap (F \circ H)$

7、 $(G \cup H) \circ F = (G \circ F) \cup (H \circ F)$

8、 $(G \cap H) \circ F \subseteq (G \circ F) \cap (H \circ F)$



§ 4.2 关系的运算

4、 $(F \circ G)^{-1} = G^{-1} \circ F^{-1}$ 的证明。

对任意 $\langle x, y \rangle \in (F \circ G)^{-1}$

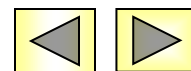
$$\Leftrightarrow \langle y, x \rangle \in F \circ G$$

$$\Leftrightarrow \exists z (\langle y, z \rangle \in G \wedge \langle z, x \rangle \in F)$$

$$\Leftrightarrow \exists z (\langle z, y \rangle \in G^{-1} \wedge \langle x, z \rangle \in F^{-1})$$

$$\Leftrightarrow \exists z (\langle x, z \rangle \in F^{-1} \wedge \langle z, y \rangle \in G^{-1})$$

$$\Leftrightarrow \langle x, y \rangle \in G^{-1} \circ F^{-1}$$



§ 4.2 关系的运算

6、 $F \circ (G \cap H) \subseteq (F \circ G) \cap (F \circ H)$ 的证明。

对任意 $\langle x, y \rangle \in F \circ (G \cap H)$

$$\Leftrightarrow \exists z (\langle x, z \rangle \in G \cap H \wedge \langle z, y \rangle \in F)$$

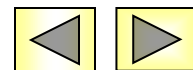
$$\Leftrightarrow \exists z (\langle x, z \rangle \in G \wedge \langle x, z \rangle \in H \wedge \langle z, y \rangle \in F)$$

$$\Leftrightarrow \exists z ((\langle x, z \rangle \in G \wedge \langle z, y \rangle \in F) \wedge (\langle x, z \rangle \in H \wedge \langle z, y \rangle \in F))$$

$$\Rightarrow \exists z (\langle x, z \rangle \in G \wedge \langle z, y \rangle \in F) \wedge \exists z (\langle x, z \rangle \in H \wedge \langle z, y \rangle \in F)$$

$$\Leftrightarrow \langle x, y \rangle \in F \circ G \wedge \langle x, y \rangle \in F \circ H$$

$$\Leftrightarrow \langle x, y \rangle \in (F \circ G) \cap (F \circ H)$$



§ 4.2 关系的运算

四、关系的幂运算

设 R 是 A 上的关系， n 为自然数，则：

$$(1) R^0 = \{ \langle x, x \rangle \mid x \in A \}$$

$$(2) R^n = R^{n-1} \circ R, n \geq 1$$



$$(1) R^m \circ R^n = R^{m+n}$$

$$(2) (R^m)^n = R^{mn}$$

其中 m, n 为自然数。



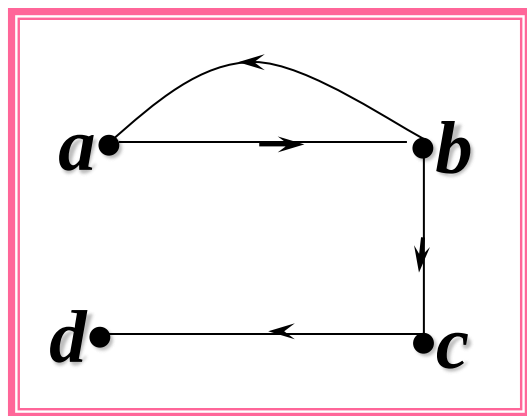
§ 4.2 关系的运算

例7: 设 $A = \{a, b, c, d\}$, A 上的一个关系

$$R = \{ \langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, d \rangle \}$$

求 R^0 , R^2 , R^3 。

解: $R^0 = \{ \langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle d, d \rangle \}$



由 R 的关系图知

$$R^2 = R \circ R = \{ \langle a, a \rangle, \langle a, c \rangle, \langle b, b \rangle, \langle b, d \rangle \}$$

$$R^3 = \{ \langle a, b \rangle, \langle a, d \rangle, \langle b, a \rangle, \langle b, c \rangle \}$$



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$$R = \{ \langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, d \rangle \}$$

求 R^0 , R^2 , R^3 。

R 的关系矩阵

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M^2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M^3 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R^2 = R \circ R = \{ \langle a, a \rangle, \langle a, c \rangle, \langle b, b \rangle, \langle b, d \rangle \}$$

$$R^3 = R^2 \circ R = \{ \langle a, b \rangle, \langle a, d \rangle, \langle b, a \rangle, \langle b, c \rangle \}$$

